Math 124 | Midterm 2 Exam | Autumn 2016

Your Name (print) ______________________________

Your Signature ______________________________

Student I.D.# | | | | | | | | **Quiz Section** | | |

• Turn in your exam when you are finished. Do not leave the room until the examination is completed. You will lose credit if you leave early.

• Turn off all electronic devices. If your phone rings accidentally, you must ask for permission to turn it off.

• This exam is closed book. You may use one 8.5x11 sheet of handwritten notes (one side), but the notes cannot include completely worked problems.

• The only allowed calculator is TI 30X-IIS.

• In order to receive credit you must show all of your work. Show enough work that the grader can determine what you did to arrive at your answers. Correct answers without justification may not receive much credit.

• If you need more room, use the backs of the pages, but **clearly** indicate you have done so.

Score

1.	(32)	
2.	(34)	
3.	(34)	
Total	(100)	

1. Find

$$\lim_{x \to \infty} \left(1 - \frac{1}{x^2}\right)^x$$

If you use L'Hospital's Rule you must explicitly show that you have checked all the hypotheses.

$$\left(1-\frac{1}{x^2}\right)^x = e^{x \ln\left(1-\frac{1}{x^2}\right)}$$

$$\frac{\ln\left(1-\frac{1}{x^2}\right) \leftarrow f}{1/x \leftarrow g} \qquad f'(x) = \frac{1}{1-1/x^2}\left(\frac{2}{x^3}\right)$$

$$g'(x) = -1/x^2$$

(1) $\ln(1-1/x^2) \to 0$ as $x \to \infty$

(2) $1/x \to 0$ as $x \to \infty$

(3) $g'(x) = -1/x^2 \neq 0$

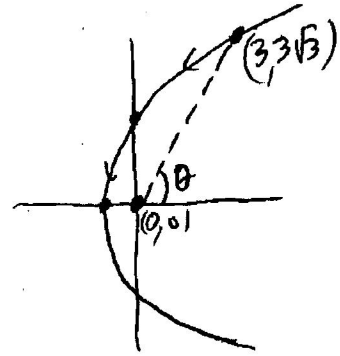
(4) $$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{1-1/x^2}\left(\frac{2}{x^3}\right)}{-1/x^2} = \frac{-2}{(1-1/x^2)x} = \frac{-2}{x-\frac{1}{x}} \to 0 \text{ as } x \to \infty$$

because $x \to \infty$, $1/x \to 0$

So by L'Hospital's Rule $\frac{f(x)}{g(x)} = \frac{\ln(1-1/x^2)}{1/x} \to 0$ as $x \to \infty$

thus $\left(1-\frac{1}{x^2}\right)^x = e^{x\ln(1-1/x^2)} \to e^0 = 1.$

2. A spacecraft flies past a planet along a hyperbola given by



$$\frac{(x+5)^2}{16} - \frac{y^2}{9} = 1.$$

The coordinates are measured in miles. A planet is located at $(0,0)$. When the spacecraft is located at $(3, 3\sqrt{3})$, the speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2} = 2$ miles per second. (the x and y coordinates are decreasing).

a) Find dx/dt and dy/dt at this point. I'll use $\dot{} = \frac{d}{dt}$

Differentiate (1) with respect to t

$$\frac{2(x+5)\dot{x}}{16} - \frac{2y\dot{y}}{9} = 0$$

at $(3,3\sqrt{3})$: $\frac{2\cdot 8}{16}\dot{x} - \frac{2\cdot 3\sqrt{3}}{9}\dot{y} = 0$ so $\dot{x} = \frac{2}{\sqrt{3}}\dot{y}$

$$\dot{x}^2 + \dot{y}^2 = 2^2$$

$$\left(\frac{2}{\sqrt{3}}\dot{y}\right)^2 + \dot{y}^2 = 4$$

$$\tfrac{4}{3}\dot{y}^2 + \dot{y}^2 = 4$$

$$\dot{y}^2 = \frac{4}{4/3+1} = \frac{12}{7}$$

$$\dot{y} = -\sqrt{\tfrac{12}{7}}$$ (negative because y decreasing)

$$\dot{x} = \frac{2}{\sqrt{3}}\left(\frac{-\sqrt{12}}{\sqrt{7}}\right) = \frac{-4}{\sqrt{7}}$$

b) How fast is the angle θ changing at this point?

$$\tan\theta = y/x$$

$$\theta = \tan^{-1}(y/x)$$

$$\dot{\theta} = \frac{1}{1+(y/x)^2}\left[\frac{x\dot{y} - y\dot{x}}{x^2}\right]$$

$$= \frac{1}{1+(\sqrt{3})^2}\left[\frac{3\left(-\sqrt{\frac{12}{7}}\right) - 3\sqrt{3}\left(-\frac{4}{\sqrt{7}}\right)}{9}\right]$$

$$= \frac{3\sqrt{3}\cdot 2}{\sqrt{7}\cdot 36} = \frac{\sqrt{3}}{6\sqrt{7}} = \frac{\sqrt{21}}{42}$$

3. Suppose that a commodity price of x dollars will result in y thousands of units sold, where x and y satisfy:

$$x + y + 2x^2y + 3xy^3 = 17,$$

For example, the point (2.00,1.000) is on the curve, which means that 1000 items are sold at \$2.00.

(a) Approximately how many units will be sold if the price is increased to \$2.05? (you must use calculus to receive credit for this question).

$$\text{①} \quad 1 + y' + 4xy + 2x^2y' + 3y^3 + 3x \cdot 3y^2y' = 0$$

$$y'(1 + 2x^2 + 9xy^2) = -1 - 4xy - 3y^3$$

at $(2,1)$: $\quad y'(1 + 8 + 18) = -1 - 8 - 3 = -12$

$$y' = \frac{-12}{27} = -4/9$$

Linear approx $\quad L(x) = 1 - 4/9(x-2)$

If $x = 2.05 \quad L(x) = 1 - 4/9(.05) = .9777\ldots$

So approx 978 units

(b) Using the second derivative test, is your estimate too low or too high?

Differentiate ① implicitly

$$y'' + 4y + 4xy' + 4xy' + 2x^2y'' + 9y^2y' + 9y^2y' + 18xy(y')^2 + 9xyy'' = 0$$

at $x=2, y=1 \quad y' = -4/9$

$$y'' + 4 + 8(-4/9) + 8\left(-\frac{4}{9}\right) + 8y'' + 9(-4/9) + 9(-4/9) + 36\left(-\frac{4}{9}\right)^2 + 18y'' = 0$$

$$y''(1+8+18) + \cancel{4} - \cancel{\frac{64}{9}} - \cancel{4} - 4 + \cancel{\frac{64}{9}} = 0$$

$$y'' = \frac{4}{27} > 0 \quad \text{at } (2,1)$$

So the graph is concave up and so the tangent line is below the curve near (2,1)

so $L(x)$ is an under-estimate near 2.

too low.